2023 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

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This is a preliminary version of the research projects, subject to change later.

0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). You may find it much easier to think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, ...$ Likewise a linear map means an F-linear, and an F-matrix means a matrix with entries or coefficients in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \{\text{linear maps } U \to V\} \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \quad \text{the } \textit{algebra} \text{ of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_n F &:= \mathrm{Aut}\, F^n \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the algebra of } n \times n \text{ matrices} \\
& \text{with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \quad \text{the identity matrix in } M_n
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be introduced and studied in class during the first two weeks.

1. Statements of Problems in Project 1

Problem 1.1.
(a) Describe all possible solutions to the matrix equation system

$$x_1^2 = x_1, \quad x_2^2 = x_2, \quad x_1 x_2 = x_2 x_1 = 0$$

in two variables in M_2, up to conjugation by Aut_2*.*
(b) Describe all those conjugation classes that satisfy the additional equation

$$x_1 + x_2 = I_2.$$

(c) Describe all possible two-sided ideals of M_2.

We saw in class that the algebra M_n itself is an M_n-space on which M_n acts by left multiplication. We also saw that an F-subspace $W \subset M_n$ is an M_n-subspace iff W is a left ideal of M_n.

Problem 1.2.
(a) Describe all possible left ideals I of M_2. Which ones of them are isomorphic to each other?
(b) Classify all minimal M_2-spaces V up to isomorphisms.
(c) Classify all M_2-spaces V up to isomorphisms.

Problem 1.3. *Generalize both Problems 1.1 and 1.2 to M_n-spaces for all n.*

2. Statements of Problems in Project 2

Let $\mathcal{U}_n$ be the set of $n \times n$ upper triangular complex matrices. As an easy exercise, you should verify that $\mathcal{U}_n$ forms a subalgebra of M_n.

Problem 2.1. *Classify all minimal* $\mathcal{U}_2$*-spaces.*

Problem 2.2. *Classify all minimal* $\mathcal{U}_3$*-spaces.*

Problem 2.3. *Classify all minimal* $\mathcal{U}_n$*-spaces, for each* n*.*